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ジョセフソン接合の電気力学を記述するモデル方程式の周期解

Periodic solutions of the model equation describing electrodynamics of the Josephson junction

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Abstract

A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. **42** (2009) 025401.

1. Model equation

1.1 Nonlocal model equation

Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x,t)$ across the Josephson junction is described by the following model equation:

$$\omega_J^{-2}\phi_{tt}+\omega_J^{-2}\eta\phi_t=-\sin\phi+\frac{\lambda_J^2}{\pi\lambda_L}\int_{-\infty}^{\infty}K_0\left(\frac{|x-x'|}{\lambda_L}\right)\phi_{x'x'}(x',t)dx'+\gamma. \qquad (1)$$

K_0 : modified Bessel function of order zero, ω_J: Josephson plasma frequency,

λ_L: London penetration depth, λ_J: Josephson penetration depth,
γ: bias current density across the junction, η: positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let l be the characteristic space scale of ϕ. When $\lambda_L << l$, then $K_0(x)\sim\pi\delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$\omega_J^{-2}\phi_{tt}+\omega_J^{-2}\eta\phi_t=-\sin\phi+\frac{\lambda_J^2}{\lambda_L}\phi_{xx}+\gamma. \qquad (2)$$

If $l << \lambda_L$, then $K_0(|x|) \sim -\ln|x|$ and Eq. (1) becomes

$$\omega_J^{-2}\phi_{tt} + \omega_J^{-2}\eta\phi_t = -\sin\phi + \frac{\lambda_J^2}{\pi\lambda_L}\int_{-\infty}^{\infty}\frac{\phi_{x'}(x',t)}{x'-x}dx' + \gamma. \qquad (3)$$

In the following, we consider the overdamped case $\eta >> 1$ and the zero bias current $\gamma = 0$. Eq. (3) can then be written in an appropriate dimensionless form as

$$\phi_t = -\sin\phi + H\phi_x, \qquad H\phi_x = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_{x'}(x',t)}{x'-x}dx'. \qquad (4)$$

1.2 Remarks

- Equation (1) is derived from Maxwell's equations combined with the London equation and the Josephson equation:
  Yu. Alief et al, Superconductivity **5** (1992) 230
  A. Gurevich, Phys. Rev. **B46** (1992) 3187.
- Equation (4) has been proposed for the first time in a purely mathematical context:
  Y. Matsuno, J. Math. Phys. **33** (1992) 3039.
- As for a review on nonlocal Josephson electrodynamics:
  A.A. Abdumalikov et al, Superconductor Science and Technology, **22** (2009) 023001
  R.G. Mints, J. Low Temp. Phys. **106** (1997) 183.

2. Exact method of solution

2.1 A nonlinear dynamical system

- Dependent variable transformation
  We seek periodic solution of (4) of the form

$$\phi = i\ln\frac{f^*}{f}, \qquad f = \prod_{j=1}^{N}\frac{1}{\beta}\sin\beta(x-x_j), \qquad (5)$$

where $x_j = x_j(t)$ are complex functions of t with $\operatorname{Im} x_j(t) > 0$, β is a positive parameter, N is an arbitrary positive integer and f^* denotes the complex conjugate expression of f. Using a formula for the Hilbert transform, one has $H\phi_x = -(\ln f^*f)_x$. Substitution of this expression and (5) into (4) gives the following bilinear equation for f and f^*

$$i(f_t^*f - f^*f_t) = \frac{i}{2}(f^2 - f^{*2}) - f_x^*f - f^*f_x. \qquad (6)$$

- A system of nonlinear ODEs for x_j

We divide (6) by f^*f, substitute f from (5) and then evaluate the residue at $x = x_j$ on both sides. This gives a system of nonlinear ODEs for x_j

$$\dot{x}_j = -\frac{1}{2\beta}\frac{\prod_{l=1}^{N}\sin\beta(x_j - x_l^*)}{\prod_{\substack{l=1\\(l\neq j)}}^{N}\sin\beta(x_j - x_l)} + i, \qquad j = 1, 2, ..., N, \tag{7}$$

where an overdot denotes differentiation with respect to t.

We introduce the following notations:

$$z = e^{2i\beta x}, \qquad \xi_j = e^{2i\beta x_j}, \qquad \eta_j = e^{2i\beta x_j^*}, \qquad j = 1, 2, ..., N, \tag{8a}$$

$$s_1 = \sum_{j=1}^{N} x_j, \qquad s_2 = \sum_{j<l}^{N} x_j x_l, \quad ..., \quad s_N = \prod_{j=1}^{N} x_j, \tag{8b}$$

$$u_1 = \sum_{j=1}^{N} \xi_j, \qquad u_2 = \sum_{j<l}^{N} \xi_j \xi_l, \quad ..., \quad u_N = \prod_{j=1}^{N} \xi_j, \tag{8c}$$

$$v_1 = \sum_{j=1}^{N} \eta_j, \qquad v_2 = \sum_{j<l}^{N} \eta_j \eta_l, \quad ..., \quad v_N = \prod_{j=1}^{N} \eta_j, \tag{8d}$$

$$t_j = \sum_{l=1}^{N} \xi_l^j, \qquad j = 1, 2, ..., N. \tag{8e}$$

In terms of $u_j (j = 1, 2, ..., N)$ and s_1, f can be written as

$$f = \frac{e^{-i\beta(Nx - s_1)}}{(2\beta i)^N}\left(z^N - u_1 z^{N-1} + u_2 z^{N-2} + ... + (-1)^N u_N\right). \tag{9}$$

Thus, u_j $(j = 1, 2, ..., N)$ and s_1 determine the function f completely.

Let us derive a system of equations for u_j. To this end, We rewrite (7) in terms of ξ_j and η_j as

$$\dot{\xi}_j = -\frac{1}{2}\alpha u_N \frac{\prod_{l=1}^{N}(\xi_j - \eta_l)}{\prod_{\substack{l=1\\(l\neq j)}}^{N}(\xi_j - \xi_l)} - 2\beta\xi_j, \qquad j = 1, 2, ..., N, \tag{10a}$$

where

$$\alpha = \prod_{j=1}^{N}(\xi_j\eta_j)^{-1/2} = e^{-i\beta(s_1 + s_1^*)}, \qquad u_N = \prod_{j=1}^{N}\xi_j = e^{2i\beta s_1}. \tag{10b}$$

Later, we show that α is a constant independent of t and u_N obeys a single nonlinear ODE.

2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables u_j defined by (8c). We multiply ξ_j^{n-1} on both sides of (10a) and sum up with respect to j from 1 to N to obtain

$$\frac{1}{n}\dot{t}_n = -\frac{\alpha}{2}u_N\sum_{s=0}^{n}(-1)^s v_s I_{n-s} - 2\beta t_n, \qquad n = 1, 2, ..., N, \tag{11a}$$

where I_{n-s} is defined by

$$I_{n-s} = \sum_{j=1}^{N}\frac{\xi_j^{N+n-s-1}}{\prod_{\substack{l=1\\(l\neq j)}}^{N}(\xi_j - \xi_l)}. \tag{11b}$$

In deriving (11), we have used the identity

$$I_n = 0, \qquad -N+1 \le n \le -1. \tag{11c}$$

• Time evolution of u_j

The time evolution of u_n follows from (11a) with the help of the formulas

$$u_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j u_j t_{n-j}, \quad 1 \le n \le N, \qquad \sum_{j=0}^{n}(-1)^j u_j I_{n-j} = 0, \quad n \ge 1, \tag{12}$$

where $u_0 = 1$ and $I_0 = 1$. In fact, differentiating the first formula in (12) by t and substituting (11a) for $\dot{t}_{n-j}$, we can show that the quantity h_n defined by

$$h_n = \dot{u}_n + \frac{\alpha}{2}u_N u_n - \frac{\alpha^{-1}}{2}u_{N-n}^* + 2\beta n u_n, \quad n = 1, 2, ..., N, \tag{13}$$

satisfies the relation

$$h_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j h_j t_{n-j} + \frac{(-1)^{n+1}r_n}{2n\alpha}, \tag{14a}$$

where

$$r_n = \sum_{j=1}^{n}u_{N-j+n}^*\left[-\sum_{s=1}^{j}(-1)^{n-s}sI_{j-s} + (-1)^{n-j}t_j\right]. \tag{14b}$$

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that $r_n \equiv 0$. It follows from this and (14a) that

$$h_n = \frac{(-1)^{n-1}}{n}\sum_{j=0}^{n-1}(-1)^j h_j t_{n-j}, \qquad n = 1, 2, ..., N. \tag{15}$$

Solving (15) with the initial condition $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$, we obtain the relations $h_n \equiv 0$ $(n = 1, 2, ..., N)$. Thus, we see that u_n evolves according to the following system of ODEs

$$\dot{u}_n + \frac{\alpha}{2}u_N u_n - \frac{\alpha^{-1}}{2}u_{N-n}^* + 2\beta n u_n = 0, \qquad n = 1, 2, ..., N. \tag{16}$$

It is remarkable that u_N obeys a single nonlinear ODE of the form

$$\dot{u}_N + \frac{\alpha}{2}u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \sqrt{\frac{u_N^*}{u_N}}, \tag{17}$$

and other $N-1$ variables $u_1, u_2, ..., u_{N-1}$ constitute a system of *linear* ODEs. Rewriting (17) in terms of s_1, we can put it into a nonlinear ODE for s_1

$$\dot{s}_1 = \frac{1}{2i\beta}\sinh(2\beta \operatorname{Im} s_1) + iN, \tag{18}$$

where $\operatorname{Im} s_1$ implies the imaginary part of s_1.

3. Periodic solutions

3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

$$\operatorname{Re}\dot{s}_1 = 0, \qquad \operatorname{Im}\dot{s}_1 = -\frac{1}{2\beta}\sinh(2\beta \operatorname{Im} s_1) + N. \tag{19}$$

Thus, the real part of s_1 becomes a constant $\operatorname{Re} s_1(t) = \operatorname{Re} s_1(0) \equiv b$ whereas integration of the equation for $\operatorname{Im} s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta \operatorname{Im} s_1$, it is given by

$$e^{-y} = \frac{2\nu_N\left(-\tanh\frac{y_0}{2} + 1\right)\cosh \nu_N t + \left\{(2\beta N + 1)\tanh\frac{y_0}{2} - 2\beta N + 1\right\}\sinh \nu_N t}{2\nu_N\left(\tanh\frac{y_0}{2} + 1\right)\cosh \nu_N t + \left\{(2\beta N - 1)\tanh\frac{y_0}{2} + 2\beta N + 1\right\}\sinh \nu_N t}, \tag{20}$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta \operatorname{Im} s_1(0)$, For $n = 1, 2, ..., N-1$, on the other hand, (16) can be written in the form

$$\dot{u}_n = -\left(\frac{1}{2}e^{-2\beta \operatorname{Im} s_1} + 2\beta n\right)u_n + \frac{\alpha^{-1}}{2}u_{N-n}^*. \tag{21}$$

Note from (10b) and $\operatorname{Re} s_1 = b$ that $\alpha = e^{-2i\beta b}$ becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

$$u_n(t) = \frac{G_n}{F}, \qquad n = 1, 2, ..., N-1, \tag{22a}$$

with

$$F = 2\nu_N \left(\tanh \frac{y_0}{2} + 1\right) \cosh \nu_N t + \left\{(2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1\right\} \sinh \nu_N t, \tag{22b}$$

$$G_n = 2\nu_N \left(\tanh \frac{y_0}{2} + 1\right) \left[u_n(0) \cosh \nu_n t \right.$$

$$\left. + \frac{1}{\nu_n} \left\{\beta(N - 2n)u_n(0) + \frac{\alpha^{-1}}{2} u_{N-n}^*(0)\right\} \sinh \nu_n t\right], \tag{22c}$$

where $\nu_n = \sqrt{\beta^2(N-2n)^2 + (1/4)}$. We see that the expression (22) with $n = N$ produces (20) and hence it can be used for all u_n.

3.2 Properties of solutions

• Asymptotic form of the solution as $t \to \infty$

$$u_n \to 0, \quad n = 1, 2, ..., N-1, \qquad u_N \to e^{2i\beta b}(\sqrt{4(\beta N)^2 + 1} - 2\beta N), \tag{23}$$

$$\phi \sim 2 \tan^{-1} \left[\frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left(Nx - b - \frac{N\pi}{2\beta}\right)\right], \tag{24}$$

$$u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta(Nx - b)}. \tag{25}$$

• Novel features of solutions

1)The asymptotic form of u does not depend on initial conditions except for a phase constant b. It represents a train of nonlinear periodic *standing* waves.

2)The initial profile of u with a spatial period π/β evolves into a periodic wave with a period $\pi/N\beta$.

3) The amplitude of the wave $A(= u_{max} - u_{min})$ is a constant independent of the wavenumber. Indeed, $u_{max} = \sqrt{4(\beta N)^2 + 1} + 1, u_{min} = \sqrt{4(\beta N)^2 + 1} - 1$ and hence $A = 2$.

4) The steady profile (25) satisfies the Peierls equation $H\phi_x = \sin \phi$ in the theory of dislocation

R. Peierls, Proc. Phys. Soc. **52** (1940) 256.

Example 1: $N=1,\ x_1(0)=3i,\ \beta=0.2$

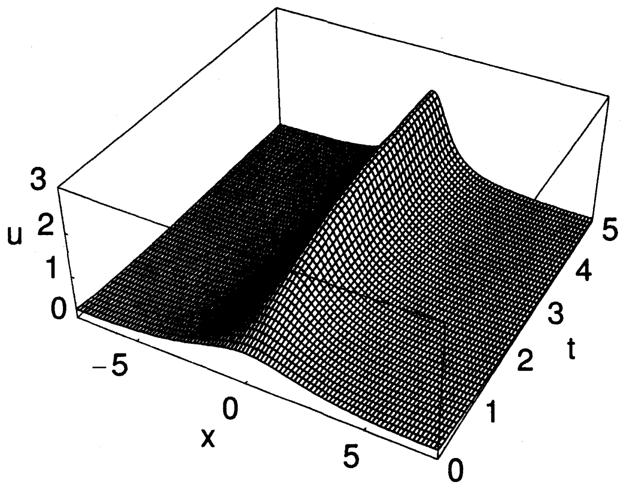

Figure 1. Time evolution of u for $N=1$ (periodic case).

Example 2: $N=2,\ x_1(0)=4i,\ x_2(0)=2i,\ \beta=0.4$

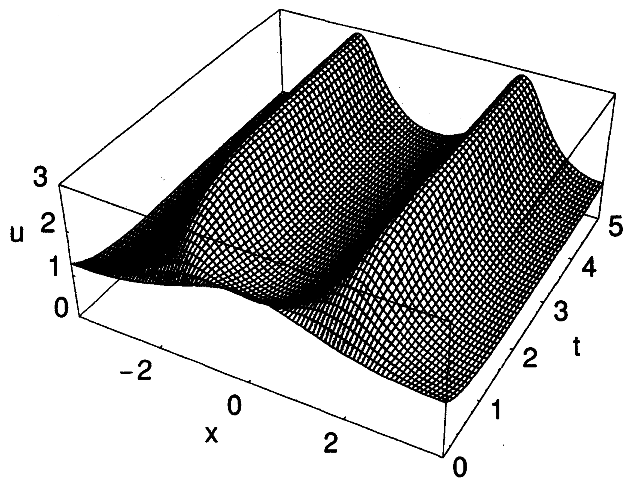

Figure 2. Time evolution of u for $N=2$ (periodic case).

3.3 Long-wave limit $\beta \to 0$

The long-wave limit $\beta \to 0$ of the periodic solutions can be derived easily. We quote the results:

$$\phi = i \ln \frac{f^*}{f}, \qquad f = \prod_{j=1}^{N}(x-x_j) = \sum_{j=0}^{N} s_j(t) x^{N-j},\ (s_0=1), \tag{26}$$

$$\dot{s}_j = -i\,\mathrm{Im}\, s_j + i(N-j+1)s_{j-1}, \quad j=1,2,...,N. \tag{27}$$

For $N = 2$, the solution reads as follows:

$$f = x^2 - s_1 x + s_2, \tag{28a}$$

$$s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1], \tag{28b}$$

$$s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2]. \tag{28c}$$

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of N Lorentzian pulses

$$u \sim \sum_{j=1}^{N} \frac{2}{(x - \sqrt{2t}\, x_{j,N})^2 + 1}, \tag{29}$$

where $x_{j,N}$ is the nth root of the Hermite polynomial of order N. These results have been detailed in Matsuno (1992).

Example 1: Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

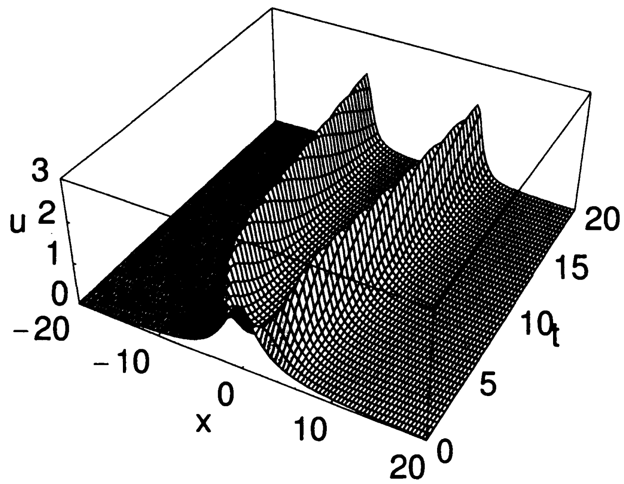

Figure 3. Time evolution of u for $N = 2$ (nonperiodic case).

4. Application

The exact method of solution developed so far can be applied to obtain periodic solutoins of the sine-Hilbert (sH) equation

$$H\theta_t = -\sin\theta, \qquad \theta = \theta(x,t). \tag{30}$$

4.1 Remark

- The sH equation was introduced by Degasperis and Santini in a purely mathematical context:

  Phys. Lett. **A 98** (1983) 240.

• The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:

J. Math. Phys. **26** (1985) 2469.

• An exact method of solution by means of biliinear transformation method was developed by Matsuno:

Phys. Lett. **A119** (1986) 229; Phys. Lett. **A120** 187(1987); J. Phys. A: Math. Gen. **20**(1987) 3587.

4.2 Periodic solutions

Here, we summarize the procedure for constructing periodic solutions of the sH equation. We seek periodic solutions of the form (5)

$$\theta = i \ln \frac{f^*}{f}, \qquad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j), \tag{31}$$

The corresponding bilinear equation for f is given by

$$(f^* f)_t = \frac{1}{2}(f^2 - f^{*2}). \tag{32}$$

The system of equations for x_j becomes

$$\dot{x}_j = \frac{1}{2i\beta} \frac{\prod_{l=1}^{N} \sin \beta(x_j - x_l^*)}{\prod_{\substack{l=1 \\ (l \neq j)}}^{N} \sin \beta(x_j - x_l)}, \quad j = 1, 2, ..., N, \tag{33}$$

and u_j satisfies the system of equations

$$\dot{u}_j = i\left(-\frac{c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^*\right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, ..., N. \tag{34}$$

The above system can be solved analytically and solutions are given explicitly.

Example: $N = 1$

Substituting $u_1 = e^{2i\beta s_1}$ into (34)

$$\dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1), \tag{35a}$$

$$\operatorname{Re} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1), \quad \operatorname{Im} \dot{s}_1 = 0, \tag{35b}$$

$$x_1 = s_1 = at + b + i\frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1) \quad b = \operatorname{Re} s_1(0), \tag{35c}$$

$$u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2} - \cos 2\beta(x - at - b)}. \tag{36}$$

Note that the solution is not a standing wave but a traveling wave.

5. Summary

- We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.

- The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.

- The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.